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WIENER INTEGRAL AND FEYNMAN INTEGRAL

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1. Introduction

Consider, for example, a classical mechanical system with Lagrangian

$$L(x, \dot{x}) = \frac{\dot{x}^2}{2} - U(x). \tag{1.1}$$

The wave function of the quantum mechanical system corresponding to this classical one changes with time t according to the Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial x^2} - U\psi, \qquad \psi(0+, x) = \varphi(x). \tag{1.2}$$

Feynman [3] expressed this wave function $\psi(t, x)$ in the following integral form, which we shall here call the *Feynman integral*

$$\psi(t, x) = \frac{1}{N}\int_{\Gamma_x} \exp\left\{\frac{i}{\hbar}\int_0^t \left[\frac{\dot{x}_\tau^2}{2} - U(x_\tau)\right] d\tau\right\} \varphi(x_t) \prod_\tau dx_\tau, \tag{1.3}$$

where Γ_x is the space of paths $X = (x_\tau, 0 < \tau \leqq t)$ with $x_0 = x$, $\prod_\tau dx_\tau$ is a uniform measure on $R^{(0,t]}$, and N is a normalization factor. It should be noted that the integral $\int_0^t [\dot{x}_\tau^2/2 - U(x_\tau)]\, d\tau$ is the classical action integral along the path X. (This idea goes back to Dirac [1].) It is easy to see that (1.3) solves (1.2) unless we require mathematical rigor. It is our purpose to define the generalized measure $\prod_\tau dx_\tau/N$, that is, the integral $\int_{\Gamma_x} F(X)\prod_\tau dx_\tau/N$, rigorously and to prove that (1.3) solves (1.2) in case $U(x) \equiv 0$ (case of no force) or $U(x) \equiv x$ (case of constant force). See theorem 5.2 and theorem 5.3 below. We hope this fact will be proved for a general $U(x)$ with some appropriate regularity conditions.

Our definition is also applicable to the *Wiener integral*; namely, using it, we shall prove that the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - Uu, \qquad u(0+, x) = f(x), \tag{1.4}$$

is given by

$$u(t, x) = \frac{1}{N}\int_{\Gamma_x} \exp\left\{-\int_0^t \left[\frac{\dot{x}^2}{2} + U(x_\tau)\right] d\tau\right\} f(x_t) \prod_\tau dx_\tau \tag{1.5}$$

for any bounded continuous function $U(x)$. See theorem 4.3. This should be called the Feynman version of Kac's theorem that

$$u(t, x) = \int_{\Gamma_x} \exp\left[-\int_0^t U(x_\tau)\, d\tau\right] f(x_t) W_x(dX) \tag{1.5'}$$

solves (1.4). In this paper the paths, that is, the points in Γ_x, are denoted with capital letters X, Y, $\cdots$ and their values at time τ are denoted with the corresponding small letters with the suffix τ such as x_τ, y_τ, $\cdots$. Now that Kac's theorem is well known to probabilists, no one bothers with its Feynman version. However, it is interesting that Kac had the Feynman version (1.5) in mind and formulated it as (1.5′) to make it rigorous [5].

Gelfand and Yaglom [4] proposed a method of defining the Feynman integral. They replaced $\hbar$ with $\hbar - i\sigma$ in (1.3) to reduce the Feynman integral to the Wiener integral and defined the Feynman integral as a limit of the Wiener integral by letting $\sigma \downarrow 0$. Our method is different from theirs in the point that we define $\prod_\tau dx_\tau/N$ directly and treat both the Feynman integral and the Wiener integral on the same level.

2. The mathematical meaning of $\prod_\tau dx_\tau/N$

What Feynman had in mind for $\prod_\tau dx_\tau$ must be a uniform measure on $R^{(0,t]}$. Rigorously speaking, this measure does not exist. Therefore, we should define it as an ideal limit of a sequence of measures on $R^{(0,t]}$. In order to be able to compute the integral (1.3) or (1.5), the approximating measures should be concentrated on the set L_x of all $X = (x_\tau, 0 < \tau \leqq t) \in R^{(0,t]}$ satisfying

(L.1) x_τ is absolutely continuous in τ,

(L.2) $\dot{x}_\tau \equiv dx_\tau/d\tau \in L^2(0, t]$,

(L.3) $\lim_{\tau \downarrow 0} x_\tau = x$.

We shall now construct a sequence of probability measures $\{P_n^{(x)}\}$ on L_x whose ideal limit is the uniform distribution on $R^{(0,t]}$. Let $\rho(\tau, \sigma)$, with $\sigma, \tau \in (0, t]$, be strictly positive definite and continuous; for example, $\rho(\tau, \sigma) = \exp(-|\tau - \sigma|)$. Let $\xi_\tau(\omega)$, $\omega \in \Omega(\mathbf{B}, P)$, be a Gaussian process with

$$E(\xi_\tau) = 0, \qquad E(\xi_\tau \xi_\sigma) = \rho(\tau, \sigma). \tag{2.1}$$

It is well known that such a Gaussian process exists. Since the continuity of $\rho(\tau, \sigma)$ implies the continuity of ξ_τ in the mean, there exists a measurable version [2] of ξ_τ. Denote that version with the same symbol ξ_τ.

Noting that

$$E\left(\int_0^t \xi_\tau^2\, d\tau\right) = \int_0^t \rho(\tau, \tau)\, d\tau < +\infty, \tag{2.2}$$

we can see that

$$P\left\{\int_0^t \xi_\tau^2\, d\tau < +\infty\right\} = 1. \tag{2.3}$$

Put

$$x_\tau^{(n)} = x + n \int_0^\tau \xi_\theta \, d\theta, \qquad 0 < \tau \leqq t. \tag{2.4}$$

Then $x_\tau^{(n)}$, for $0 < \tau \leqq t$, is also a Gaussian process with

$$\begin{aligned} E[x_\tau^{(n)}] &= x, \\ E\{[x_\sigma^{(n)} - x][x_\sigma^{(n)} - x]\} &= n^2 \int_0^\tau \int_0^\sigma \rho(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2. \end{aligned} \tag{2.5}$$

Denote by $P_n^{(x)}$ the probability distribution of the sample function $X^{(n)}$ of the process $x_\tau^{(n)}$, with $0 < \tau \leqq t$. Then $P_n^{(x)}$ is concentrated on L_x and any finite dimensional marginal distribution of $P_n^{(x)}$, say over coordinates $\tau_1, \tau_2, \cdots, \tau_m$, is Gaussian with thc density

$$\frac{b^{1/2}}{(2\pi n^2)^{m/2}} \exp\left[\frac{-1}{2n^2} \sum_{i,j=1}^m b_{ij}(x_i - x)(x_j - x)\right], \tag{2.6}$$

where the matrix (b_{ij}) is the inverse of the matrix (v_{ij}) with

$$v_{ij} = \int_0^{\tau_i} \int_0^{\tau_j} \rho(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2, \qquad i, j = 1, 2, \cdots, m \tag{2.7}$$

and b is the determinant of (b_{ij}). The existence of (b_{ij}), that is, the nonvanishing of the determinant of (v_{ij}) results from the assumption that $\rho(\tau, \sigma)$ is strictly positive definite.

Since the Gaussian distribution (2.6) tends to a uniform distribution on the m-space in the sense that, for any almost periodic function $f(x_1, x_2, \cdots, x_m)$,

(2.8)
$$\int \cdots \int f(x_1, \cdots, x_m) \frac{b^{1/2}}{(2\pi n^2)^{m/2}} \exp\left[\frac{-1}{2n^2} \sum_{i,j=1}^m b_{ij}(x_i - x)(x_j - x)\right] dx_1 \cdots dx_m$$

tends to the Bohr mean $\mathfrak{M}(f)$ of f as $n \to \infty$, it is reasonable to say that $P_n^{(x)}$, for $n = 1, 2, \cdots$ approximates the uniform distribution on $R^{(0,t]}$ and that $\prod_\tau dx_\tau$ is an ideal limit of this sequence.

N must be also an ideal limit of a sequence of numbers $\{N_n\}$ such that $P_n^{(x)}/N_n$ tends to $\prod_\tau dx_\tau/N$ in some sense.

Keeping these heuristic considerations in mind, we shall give a mathematical meaning to $\prod_\tau dx_\tau/N$, that is, to the linear functional $I(F) = \int F(X) \prod_\tau dx_\tau/N$. There are many ways of defining this functional in accordance with the choice of the sequence $\{N_n\}$. We shall express $I(F)$ as $I(F, N_n)$ referring to the sequence $\{N_n\}$.

Definition.

$$I(F, N_n) = \lim_{n \to \infty} \frac{1}{N_n} \int_{L_x} F(X) P_n^{(x)}(dX). \tag{2.9}$$

The domain $\mathfrak{D}(N_n)$ of this functional $I(F, N_n)$ is the set of all F for which the limit in (2.9) exists and is finite.

Fixing $\{N_n\}$, we shall write $\mathfrak{D}(N_n)$ as $\mathfrak{D}$ and $I(F, N_n)$ as

$$(2.10) \qquad \frac{1}{N}\int_{\Gamma_x} F(X) \prod_{\tau} dx_\tau.$$

We shall mention three interesting cases.

(i) *Uniform integral.* $N_n = 1$, with $n = 1, 2, \cdots$. If $F(X)$ is of the form $f(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_m})$ with an almost periodic function $f(x_1, x_2, \cdots, x_m)$, then $F \in \mathfrak{D}$ and

$$(2.11) \qquad \frac{1}{N}\int_{\Gamma_x} F(X) \prod_{\tau} dx_\tau = \mathfrak{M}(f),$$

where $\mathfrak{M}(f)$ is the Bohr mean of f.

(ii) *Wiener integral.* $N_n = 1/\prod_\nu(1 + n^2\lambda_\nu)^{1/2}$, $n = 1, 2, \cdots$, where λ_ν will be defined in the next section. We shall discuss the Wiener integral in section 4.

(iii) *Feynman integral.* $N_n = 1/\prod_\nu(1 + n^2\lambda_\nu/\hbar i)^{1/2}$, with $n = 1, 2, \cdots$, with the same λ_ν as in (ii). This will be discussed in section 5.

3. Orthogonalization method

In the following sections we shall be faced with the integrals of the type

$$(3.1) \qquad I = \int_{L_x} G(X) P_n^{(x)}(dX).$$

Recalling that $P_n^{(x)}$ was defined as the probability distribution of the sample path $X^{(n)}$ of the process

$$(3.2) \qquad x_\tau^{(n)}(\omega) = x + n \int_0^\tau \xi_\theta(\omega)\, d\theta$$

introduced in section 2, the integral I can be expressed as the mean value of $G[X^{(n)}(\omega)]$ on $\Omega(\mathbf{B}, P)$

$$(3.3) \qquad I = \int_\Omega G[X^{(n)}(\omega)]P(d\omega) \equiv E\{G[X^{(n)}]\}.$$

To compute this, we shall use the usual *orthogonalization method.* The idea is as follows. Let T denote the operator from $L^2(0, t]$ into itself,

$$(3.4) \qquad (T\eta)(\tau) = \int_0^t \rho(\tau, \sigma)\eta(\sigma)\, d\sigma.$$

Then T is a strictly positive-definite compact operator. Therefore T has positive eigenvalues $\{\lambda_\nu\}$ whose eigenfunctions $\{\eta_\nu\}$ constitute a complete orthonormal system in $L^2(0, t]$.

Now put

$$(3.5) \qquad a_\nu(\omega) = (\xi, \eta_\nu) = \int_0^t \xi_\tau(\omega)\eta_\nu(\tau)\, d\tau;$$

this inner product is well defined, thanks to (2.3). Then $\{a_\nu\}$ is a Gaussian system, since ξ_τ is a Gaussian process. Equation (2.1) implies that

$$E(a_\nu a_\mu) = \int_0^t \int_0^t \rho(\tau, \sigma)\eta_\nu(\tau)\eta_\mu(\sigma)\, d\tau\, d\sigma. \tag{3.6}$$

Therefore a_ν, with $\nu = 1, 2, \cdots$, are independent and each a_ν is Gaussian with mean 0 and variance λ_ν. Since we have

$$\sum_\nu \lambda_\nu = \sum_\nu E(a_\nu^2) = E(\sum_\nu a_\nu^2) = E\left(\int_0^t \xi_\tau^2\, d\tau\right) = \int_0^t \rho(\tau, \tau)\, d\tau, \tag{3.7}$$

the continuity of $\rho(\tau, \sigma)$ implies

$$\sum_\nu \lambda_\nu < +\infty; \tag{3.8}$$

this fact will be useful in the following sections.

Noting that

$$\xi_\tau(\omega) = \sum_\nu a_\nu(\omega)\eta_\nu(\tau) \tag{3.9}$$

and that

$$x_\tau^{(n)}(\omega) = x + n \int_0^\tau \xi_\theta(\omega)\, d\theta. \tag{3.10}$$

we can express I in the form

$$I = E\{H(a_1, a_2, \cdots)\} \tag{3.11}$$

with some H. Using the independence and the normality of $\{a_\nu\}$, we can compute (3.11) more easily than the original form (3.1).

4. Wiener integral

We shall now discuss the integral (2.10) for

$$N_n = \frac{1}{\prod_\nu (1 + n^2\lambda_\nu)^{1/2}}, \qquad n = 1, 2, \cdots, \tag{4.1}$$

where λ_ν, with $\nu = 1, 2, \cdots$, are the eigenvalues introduced in section 3.

We can verify easily the convergence of the infinite sums and infinite products appearing in this section by appealing to (3.8).

LEMMA 4.1.

$$Q_n^{(x)}(dX) \equiv \frac{1}{N_n} \exp\left(-\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau\right) P_n^{(x)}(dX) \tag{4.2}$$

is a probability distribution on L_x.

PROOF. Using the orthogonalization method, we get

$$\begin{aligned} I_n &= \int_{L_x} \exp\left(-\int_0^t \frac{\dot{x}^2}{2}\, d\tau\right) P_n^{(x)}(dX) \\ &= E\left[\exp\left(-n^2 \int_0^t \frac{\xi_\tau^2}{2}\, d\tau\right)\right] \\ &= E\left[\exp\left(-n^2 \sum_\nu \frac{a_\nu^2}{2}\right)\right]. \end{aligned} \tag{4.3}$$

Noting that the a_ν, for $\nu = 1, 2, \cdots$, are independent, we have

$$(4.4)\qquad \begin{aligned} I_n &= \prod_\nu E\left[\exp\left(-\frac{n^2 a_\nu^2}{2}\right)\right] \\ &= \prod_\nu \int_{-\infty}^{+\infty} \frac{1}{(2\pi\lambda_\nu)^{1/2}} \exp\left(\frac{-\alpha^2}{2\lambda_\nu} - \frac{n^2\alpha^2}{2}\right) d\alpha \\ &= \prod_\nu \frac{1}{(1+n^2\lambda_\nu)^{1/2}} \\ &= N_n, \end{aligned}$$

which proves our lemma.

LEMMA 4.2. *For any* $g \in L^2(0, t]$, *we have*

$$(4.5)\qquad \int_{L_x} \exp\left[i \int_0^t \dot{x}_\tau g(\tau)\, d\tau\right] Q_n^{(x)}(dX) = \exp\left[-\sum_\nu \frac{n^2\lambda_\nu g_\nu^2}{2(n^2\lambda_\nu + 1)}\right],$$

where $g_\nu = (g, \eta_\nu) \equiv \int_0^t g(\tau)\eta_\nu(\tau)\, d\tau$, *for* $\nu = 1, 2, \cdots$.

PROOF. By the same idea as in lemma 1, we have

$$(4.6)\qquad \begin{aligned} I_n &= \int_{L_x} \exp\left[i \int_0^t \dot{x}_\tau g(\tau)\, d\tau\right] Q_n^{(x)}(dX) \\ &= \frac{1}{N_n} \int_{L_x} \exp\left[i \int_0^t \dot{x}_\tau g(\tau)\, d\tau - \int_0^t \frac{\dot{x}_\tau^2}{2} d\tau\right] P_n^{(x)}(dX) \\ &= \frac{1}{N_n} E\left[\exp\left(in \sum_\nu g_\lambda a_\nu - n^2 \sum_\nu \frac{a_\nu^2}{2}\right)\right] \\ &= \frac{1}{N_n} \prod_\nu E\left[\exp\left(ing_\nu a_\nu - n^2 \frac{a_\nu^2}{2}\right)\right] \\ &= \frac{1}{N_n} \prod_\nu \int \frac{1}{(2\pi\lambda_\nu)^{1/2}} \exp\left(\frac{-\alpha^2}{2\lambda_\nu} + ing_\nu\alpha - \frac{n^2\alpha^2}{2}\right) d\alpha \\ &= \frac{1}{N_n} \prod_\nu \frac{1}{(1+n^2\lambda_\nu)^{1/2}} \exp\left[\frac{-n^2\lambda_\nu g_\nu^2}{2(n^2\lambda_\nu + 1)}\right], \end{aligned}$$

which proves (4.5) by virtue of (4.1).

As an immediate result from lemma 4.2, we obtain the following lemma, noting that $\sum_\nu g_\nu^2 = \int_0^t g(\tau)^2\, d\tau$ and that $\int_0^t g(\tau)\, dB(\tau)$ is Gaussian distributed with mean 0 and variance $\int_0^t g(\tau)^2\, d\tau$ for the Brownian motion $B(\tau)$.

LEMMA 4.3. *For any* $g \in L^2(0, t]$,

$$(4.7)\qquad \exp\left[i \int_0^t \dot{x}_\tau g(\tau)\, d\tau - \int_0^t \frac{\dot{x}_\tau^2}{2} d\tau\right] \in \mathfrak{D},$$

and

$$\frac{1}{N}\int_{L_x}\exp\left[i\int_0^t \dot{x}_\tau g(\tau)\,d\tau - \int_0^t \frac{\dot{x}_\tau^2}{2}\,d\tau\right]\prod_\tau dx_\tau = \int_{L_x}\exp\left[i\int_0^t g(\tau)\,dx_\tau\right]W_x(dX), \tag{4.8}$$

where W_x is the probability measure for the Brownian motion process starting at x, namely the Wiener measure.

THEOREM 4.1.

$$\frac{1}{N}\exp\left(-\int_0^t \frac{\dot{x}_\tau^2}{2}\,d\tau\right)\prod_\tau dx_\tau = W_x(dX); \tag{4.9}$$

rigorously speaking, we have, for any continuous bounded tame function $F(X)$,

$$F(X)\exp\left(-\int_0^t \frac{\dot{x}_\tau^2}{\tau}\,d\tau\right) \in \mathfrak{D} \tag{4.10}$$

$$\frac{1}{N}\int_{\Gamma_x} F(X)\exp\left(-\int_0^t \frac{\dot{x}_\tau^2}{2}\,d\tau\right)d\prod_\tau x_\tau = \int_{\Gamma_x} F(X)W_x(dX). \tag{4.11}$$

A tame function is a function defined on an infinite dimensional space which depends only on a finite number of coordinates.

PROOF. $F(X)$ can be expressed as

$$F(X) = f(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_m}), \quad 0 < \tau_1 < \cdots < \tau_m \leqq t, \tag{4.12}$$

with a continuous bounded function f of m real variables. To obtain theorem 4.1, it is enough to prove that

$$\lim_{n\to\infty}\int_{L_x} f(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_m})Q_n^{(x)}(dX) = \int_{\Gamma_x} f(x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_m})W_x(dX). \tag{4.13}$$

Let $\tilde{Q}_n^{(n)}$ and $\tilde{W}_x$ denote the marginal distributions of $Q_n^{(x)}$ and W_x over the coordinates $\tau_1, \tau_2, \cdots, \tau_n$ respectively. Then

$$\begin{aligned} I_n &= \int\cdots\int \exp\left[i(z_1\alpha_1 + \cdots + z_m\alpha_m)\right]\tilde{Q}_n^{(x)}(d\alpha_1\cdots d\alpha_m) \\ &= \int_{L_x}\exp\left[i(z_1x_{\tau_1} + \cdots + z_mx_{\tau_m})\right]Q_n^{(x)}(dX) \\ &= \exp\left[i(z_1 + \cdots + z_m)x\right]\int_{L_x}\exp\left[i\int g(\tau)\dot{x}_\tau\,d\tau\right]Q_n^{(x)}(dX), \end{aligned} \tag{4.14}$$

where $g(\tau) = \sum_{j=1}^m z_j\varphi_j(\tau)$ and $\varphi_j(\tau)$ is the indicator function of the set $(0, \tau_j)$. Using lemma 4.3, we get

$$\begin{aligned} I_n &\to \int_{\Gamma_x}\exp\left[i(z_1 + \cdots + z_m)x + i\int g(\tau)\,dx_\tau\right]W_x(dX) \\ &= \int_{\Gamma_x}\exp\left[i(z_1x_{\tau_1} + \cdots + z_mx_{\tau_m})\right]W_x(dX) \\ &= \int\cdots\int\exp\left[i(z_1\alpha_1 + \cdots + z_m\alpha_m)\right]\tilde{W}_x(d\alpha_1\,d\alpha_2\cdots d\alpha_n). \end{aligned} \tag{4.15}$$

Therefore $\tilde{Q}_n^{(x)} \to \tilde{W}_x$ in the weak sense as $n\to\infty$, which implies (4.13).

THEOREM 4.2. *If $f(x)\colon R^1 \to C$ is continuous and bounded, then*

$$\exp\left(-\int_0^t \frac{\dot{x}_\tau^2}{2}\,d\tau\right) f(x_t) \in \mathfrak{D} \tag{4.16}$$

and

$$u(t, x) \equiv \frac{1}{N} \int_{\Gamma_x} \exp\left(-\int_0^t \frac{\dot{x}_\tau^2}{2}\,d\tau\right) f(x_t) \prod_\tau dx_\tau \tag{4.17}$$

solves

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad u(0+, x) = f(x). \tag{4.18}$$

PROOF. Using the previous theorem, we have

$$u(t, x) = \int_{\Gamma_x} f(x_t) W_x(dX) = \int f(y) \frac{1}{(2\pi t)^{1/2}} \exp\left[-\frac{(x-y)^2}{2t}\right] dy \tag{4.19}$$

and this solves (4.18).

THEOREM 4.3. (*Feynman's version of Kac's theorem.*) *If $f(x)$ and $U(x)$ are continuous and bounded, then*

$$\exp\left\{-\int_0^t \left[\frac{\dot{x}_\tau^2}{2} + U(x_\tau)\right] d\tau\right\} f(x_t) \in \mathfrak{D} \tag{4.20}$$

and

$$u(t, x) = \frac{1}{N} \int_{\Gamma_x} \exp\left\{-\int_0^t \left[\frac{\dot{x}_\tau^2}{2} + U(x_\tau)\right] d\tau\right\} f(x_t) \prod_\tau dx_\tau \tag{4.21}$$

solves

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - U(x)u, \qquad u(0+, x) = f(x). \tag{4.22}$$

PROOF. It is enough, by virtue of Kac's theorem, to prove that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{N_n} \int_{L_x} \exp\left\{-\int_0^t \left[\frac{\dot{x}_\tau^2}{2} + U(x_\tau)\right] d\tau\right\} f(x_t) P_n(dX) \\ = \int_{\Gamma_x} \exp\left[-\int_0^t U(x_\tau)\,d\tau\right] f(x_t) W_x(dX).\end{aligned} \tag{4.23}$$

Denoting the integrals in (4.23) by I_n and I respectively, we obtain

$$\begin{aligned} I_n &= \int_{L_x} \exp\left[-\int_0^t U(x_\tau)\,d\tau\right] f(x_t) Q_n^{(x)}(dX) \\ &= I_{nm} + R_{nm},\end{aligned} \tag{4.24}$$

where

$$I_{nm} = \sum_{\nu=1}^{m} \frac{(-1)^\nu}{\nu!} \int_0^t \cdots \int_0^t \int_{L_x} U(x_{\tau_1}) \cdots U(x_{\tau_\nu}) Q_n^{(x)}(dX)\, d_{\tau_1} \cdots d\tau_\nu, \tag{4.25}$$

$$|R_{nm}| \leq \frac{t^{m+1}}{(m+1)!} \|U\|_\infty^{m+1} \|f\|_\infty \exp(\|U\|_\infty t), \tag{4.26}$$

where $\| \;\; \|_\infty$ means the uniform norm. Therefore I_{nm} tends to I_n uniformly in n as $m \to \infty$. Using theorem 4.1, we have

$$(4.27)\qquad I_{nm} \to I_{\infty m} \equiv \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!} \int_0^t \cdots \int_0^t \int_{L_x} U(x_{\tau_1}) \cdots U(x_{\tau_\nu}) f(x_t) W_x(dX)\, d\tau_1 \cdots d\tau_\nu$$

and it is easy to see that $I_{\infty m} \to I$ as $n \to \infty$. Taking the uniform convergence of $\lim_{m\to\infty} I_{nm} = I_n$ into account, we have

$$(4.28)\qquad \lim_{n\to\infty} I_n = \lim_{n\to\infty} \lim_{m\to\infty} I_{nm} = \lim_{m\to\infty} \lim_{n\to\infty} I_{nm} = \lim_{m\to\infty} I_{\infty m} = I,$$

which completes our proof.

5. Feynman integral

In this section we shall discuss (2.10) for

$$(5.1)\qquad N_n = \frac{1}{\prod_\nu \left(1 + \frac{n^2\lambda_\nu}{\hbar i}\right)^{1/2}}, \qquad n = 1, 2, \cdots.$$

As in section 4, we can easily verify the convergence of the infinite sums and infinite products, by appealing to (3.8).

LEMMA 5.1. *If* $\mathrm{Re}(b) > 0$ *and* c *is real, then*

$$(5.2)\qquad \int_{-\infty}^{\infty} \exp(-b\alpha^2 + ic)\, d\alpha = \left(\frac{\pi}{b}\right)^{1/2} \exp\left(-\frac{c^2}{4b}\right).$$

PROOF. This is true if $b > 0$. By analytic continuation, we can verify (5.2) for $\mathrm{Re}(b) > 0$.

LEMMA 5.2. *If* $g(\tau) \in L^2(0, t]$, *then*

$$(5.3)\qquad \frac{1}{N_n} \int_{L_x} \exp\left[\frac{i}{\hbar} \int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau + i \int_0^t x_\tau g(\tau)\, d\tau\right] P_n^{(n)}(dX) = \exp\left[-\sum_\nu \frac{n^2\lambda_\nu \hbar i g_\nu}{2(n^2\lambda_\nu + \hbar i)}\right],$$

where

$$(5.4)\qquad g_\nu = (g, \eta_\nu) = \int_0^t g(\tau)\eta_\nu(\tau)\, d\tau.$$

PROOF. We shall use the orthogonalization method introduced in section 3.

$$(5.5)\qquad \begin{aligned} I_n &= \frac{1}{N_n} \int_{L_x} \exp\left[\frac{i}{\hbar} \int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau + i \int_0^t \dot{x}_\tau g(\tau)\, d\tau\right] P_n^{(x)}(dX) \\ &= \frac{1}{N_n} E\left\{\exp\left[\frac{in^2}{2\hbar} \sum_\nu (a_\nu^2 + ing_\nu a_\nu)\right]\right\} \\ &= \frac{1}{N_n} \prod_\nu E\left[\exp\left(\frac{in^2}{2\hbar} a_\nu^2 + ing_\nu a_\nu\right)\right] \\ &= \frac{1}{N_n} \prod_\nu \int_{-\infty}^{\infty} \frac{1}{(2\pi\lambda_\nu)^{1/2}} \exp\left(-\frac{\alpha^2}{2\lambda_\nu} + \frac{in^2\alpha^2}{2\hbar} + ing_\nu a_\nu\right) d\alpha. \end{aligned}$$

Using lemma 5.1 to evaluate this integral, we have

$$I_n = \exp\left[-\sum_\nu \frac{n^2\lambda_\nu \hbar i g_\nu^2}{2(n^2\lambda_\nu + \hbar i)}\right]. \tag{5.6}$$

Noting that $\sum g_\nu^2 = \int_0^t g(\tau)^2\, d\tau$, we obtain, as an immediate result from lemma 5.2,

THEOREM 5.1. *If $g(\tau) \in L^2(0, t]$, then*

$$\exp\left[\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau + i\int_0^t \dot{x}_\tau g(\tau)\, d\tau\right] \in \mathfrak{D} \tag{5.7}$$

and

$$\frac{1}{N}\int_{\Gamma_x} \exp\left[\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau + i\int_0^t \dot{x}_\tau g(\tau)\, d\tau\right]\prod_\tau dx_\tau = \exp\left[-\hbar i\int_0^t \frac{g^2(\tau)}{2}\, d\tau\right]. \tag{5.8}$$

THEOREM 5.2. *If the Fourier transform of $\varphi(x)$ is a continuous function with compact support, then*

$$\exp\left(\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau\right)\varphi(x_\tau) \in \mathfrak{D}, \tag{5.9}$$

and

$$\psi(t, x) = \frac{1}{N}\int_{\Gamma_x} \exp\left(\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau\right)\varphi(x_t)\prod_\tau dx_\tau \tag{5.10}$$

solves

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial x^2}, \qquad \psi(0, x) = \varphi(x). \tag{5.11}$$

PROOF. It is enough to prove that

$$I_n = \frac{1}{N_n}\int_{L_x} \exp\left(\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau\right)\varphi(x_t) P_n^{(x)}(dX) \tag{5.12}$$

tends to

$$\int \frac{1}{(2\pi\hbar i t)^{1/2}} \exp\left[\frac{-(x-y)^2}{2\hbar i t}\right]\varphi(y)\, dy. \tag{5.13}$$

Denoting the Fourier transform of $\varphi(x)$ by $\hat{\varphi}(\hat{x})$ or $(\mathfrak{F}\varphi)(x)$ as

$$\hat{\varphi}(\hat{x}) = (\mathfrak{F}\varphi)(\hat{x}) = \int_{-\infty}^{\infty} \exp(2\pi i\hat{x}x)\varphi(x)\, dx, \tag{5.14}$$

we have

$$\begin{aligned} I_n &= \frac{1}{N_n}\int_{L_x} \exp\left(\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau\right)\int_{-\infty}^{\infty} \exp(-2\pi i\hat{x}x_t)\hat{\varphi}(x)\, d\hat{x}\, P_n^{(x)}(dX) \\ &= \frac{1}{N_n}\int_{-\infty}^{\infty} \hat{\varphi}(\hat{x})\exp(-2\pi i\hat{x}x)\, d\hat{x}\int_{L_x} \exp\left(\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau \right. \\ &\qquad\qquad \left. - 2\pi i\hat{x}\int_0^t \dot{x}_\tau\, d\tau\right) P_n^{(x)}(dX). \end{aligned} \tag{5.15}$$

Putting $g(t) = -2\pi\hat{x}\dot{x}_\tau$ in lemma 5.2 to compute this integral over L_x, we have

$$I_n = \int_{-\infty}^{\infty} \varphi(\hat{x}) \exp\left[-2\pi i\hat{x}x - \sum_\nu \frac{n^2\hbar i\lambda_\nu g_\nu^2}{2(n^2\lambda_\nu + \hbar i)}\right] dx. \tag{5.16}$$

Recalling the assumption that $\hat{\varphi}(\hat{x})$ has a compact support, and noting that

$$\sum_\nu g_\nu^2 = \int_0^t g^2(\tau)\, d\tau = 4\pi^2\hat{x}^2 t, \tag{5.17}$$

we have

$$\lim_{n\to\infty} I_n = \int_{-\infty}^{\infty} \hat{\varphi}(\hat{x}) \exp(-2\pi i\hat{x}x - 2\pi^2\hbar it\hat{x}^2)\, d\hat{x}. \tag{5.18}$$

Since the Fourier transform of

$$N(x, \hbar it) = \frac{1}{(2\pi\hbar it)^{1/2}} \exp\left(\frac{-x^2}{2\hbar it}\right) \tag{5.19}$$

in the Schwartz distribution sense [6] is $\exp(-2\pi^2\hbar it\hat{x}^2)$, we obtain

$$\begin{aligned} \lim_{n\to\infty} I_n &= \mathfrak{F}^{-1}\{(\mathfrak{F}\varphi)\mathfrak{F}[N(\cdot, \hbar it)]\} \\ &= \varphi(x) * N(x, \hbar it), \end{aligned} \tag{5.20}$$

namely

$$\psi(t, x) = N(x, \hbar it) * \varphi(x), \tag{5.21}$$

which completes our proof.

THEOREM 5.3. *If the Fourier transform of $\varphi(x)$ has compact support, then we have*

$$\exp\left[\frac{i}{\hbar}\int_0^t \left(\frac{\dot{x}_\tau^2}{2} - x_\tau\right) d\tau\right] \varphi(x_t) \in \mathfrak{D} \tag{5.22}$$

and

$$\psi(t, x) = \frac{1}{N}\int_{\Gamma_x} \exp\left[\frac{i}{\hbar}\int_0^t \left(\frac{\dot{x}_\tau^2}{2} - x_\tau\right) d\tau\right] \varphi(x_t) \prod_\tau dx_\tau \tag{5.23}$$

solves

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2} - x\psi, \qquad \psi(0+, x) = \varphi(x). \tag{5.24}$$

PROOF. Defining $\hat{\varphi}(\hat{x})$ as in (5.14), we obtain

$$\begin{aligned} I_n &= \frac{1}{N_n}\int_{L_x} \exp\left[\frac{i}{\hbar}\int_0^t \left(\frac{\dot{x}_\tau^2}{2} - x_\tau\right) d\tau\right] \varphi(x_t) P_n^{(x)}(dX) \\ &= \frac{1}{N_n}\int \hat{\varphi}(\hat{x})\, d\hat{x} \int_{L_x} \exp\left(\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau - \frac{i}{\hbar}\int_0^t x_\tau\, d\tau - 2\pi i\hat{x}x_t\right) P_n^{(x)}(dX) \\ &= \frac{1}{N_n}\int \hat{\varphi}(\hat{x}) \exp\left[-2\pi i\left(\hat{x} + \frac{t}{2\pi\hbar}\right)x\right] d\hat{x} \int_{L_x} \exp\left[\frac{i}{\hbar}\int_0^t \frac{\dot{x}_\tau^2}{2}\, d\tau \right. \\ &\qquad\qquad \left. + i\int_0^t g(\tau)\dot{x}_\tau\, d\tau\right] P_n^{(x)}(dX), \end{aligned} \tag{5.25}$$

where $g(\tau) = -(t-\tau)/\hbar - 2\pi\hat{x}$. Using lemma 5.2 to evaluate this integral over L_x and recalling that $\hat{\varphi}$ has compact support to take the limit as $n \to \infty$, we have

$$\begin{aligned}(5.26)\quad \psi(t, x) &= \int \hat{\varphi}(\hat{x}) \exp\left[-2i\left(\hat{x} + \frac{t}{2\pi\hbar}\right)x - \frac{\hbar i}{2}\int_0^t \left(-\frac{t-\tau}{\hbar} - 2\pi\hat{x}\right)^2 d\tau\right] d\hat{x} \\ &= \int \hat{\varphi}\left(\hat{x} - \frac{t}{2\pi\hbar}\right) \exp\left[-2\pi i\hat{x}x - \frac{\hbar i}{2}\int_0^t \left(\frac{\tau}{\hbar} - 2\pi\hat{x}\right)^2 d\tau\right] d\hat{x} \\ &= \int \hat{\varphi}\left(\hat{x} - \frac{t}{2\pi\hbar}\right) \exp\left\{-2\pi i\hat{x}x - \frac{\hbar^2 i}{6}\left[\left(-2\pi\hat{x} + \frac{t}{\hbar}\right)^3 - (-2\pi\hat{x})^3\right]\right\} d\hat{x}.\end{aligned}$$

Thus we get

$$\begin{aligned}(5.27)\qquad \hat{\psi}(t, \hat{x}) &\equiv \mathfrak{F}[\psi(t, \cdot)] \\ &= \hat{\varphi}\left(\hat{x} - \frac{t}{2\pi\hbar}\right) \exp\left\{-\frac{\hbar^2 i}{6}\left[\left(-2\pi\hat{x} + \frac{t}{\hbar}\right)^3 - (-2\pi\hat{x})^3\right]\right\}.\end{aligned}$$

Simple computation shows that $\hat{\psi}(t, \hat{x})$ satisfies

$$(5.28)\qquad \frac{\hbar}{i}\frac{\partial\hat{\psi}}{\partial t} = \frac{\hbar^2}{2}(-2\pi i\hat{x})^2\hat{\psi} - \frac{1}{2\pi i}\frac{\partial\hat{\psi}}{\partial\hat{x}}, \qquad \hat{\varphi}(0+, \hat{x}) = \hat{\varphi}(\hat{x}).$$

This implies (5.13).

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